JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 25, No. 3, May–June 2002

# Engineering Notes

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## Increasing the Numerical Robustness of Balanced Model Reduction

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### I. Introduction

PROHIBITIVELY large models need to be reduced for analysis and controller synthesis because large-order models result in 1) numerical robustness and computational difficulties, 2) high computational cost for analysis, and 3) large-order controllers. Balanced reduction is an accepted method for reducing the order of a model.[1,2] Ironically, the computation of a balanced realization for the large-order model can be computationally prohibitive and numerically unstable. In this Note a numerically stable computation of the balanced realization is developed and demonstrated on a large-order finite element method (FEM) model of the Space Interferometery Mission (SIM) spacecraft. The proposed method derives its benefit by simultaneously balancing and truncating, in contrast to the more standard balancing followed by truncating.

### II. Standard Balanced Reduction Algorithm

Balanced realizations transform the system to normalize the influence of the inputs and outputs on the system states. In the balanced system states that are marginally observable will be marginally controllable, and they can be truncated to reduce the order of the system. We examine the procedure for balancing the linear time-invariant system represented in state-space form,

$$\dot{x} = Ax + Bu, \qquad y = Cx \tag{1}$$

as presented by Zhou et al.[3] to determine shortcomings of balanced reduction and because it forms the foundation of our balanced reduction algorithm. $x$, $u$, and $y$ are the state, input, and output vectors, respectively, and $A$, $B$, and $C$ are the dynamics, input and output matrices. Alternate balancing algorithms exist based on lower-triangular and upper-triangular or Cholesky factorizations. We have been unable to balance/reduce many lightly damped spacecraft models with standard balanced reduction algorithms. This limitation has motivated the numerical robustness improvements detailed in this Note.

We assume the system is stable and minimal and solve

$$AL_c + L_cA^T + BB^T = 0 \tag{2}$$

for the controllability gramian $L_c$ and

$$A^TL_o + L_oA + C^TC = 0 \tag{3}$$

for the observability gramian $L_o$.

If the system is minimal and stable, the unique gramians can be found and are both positive definite. For large-order flexible systems numerical problems can make the computation on the gramians difficult. A common algorithm uses a Schur decomposition to transform the system into a triangular form and builds up the solution by solving equation at a time, by substituting solutions of equations with one unknown into equations with two unknowns, and so on, until the triangular system is solved. This solution technique allows error to accumulate, and flexible systems are particularly sensitive. The numerically robust balanced reduction algorithm presented here does not address these difficulties but assumes Eqs. (2) and (3) can be solved. To track possible numeric problems, a determination of the maximum singular value of the residual, when the computed "solution" is substituted into to Eqs. (2) and (3), is recommended.

We use a singular value decomposition on $L_c$ to write

$$L_c = U_c\Lambda_cU_c^T \tag{4}$$

The elements of diagonal matrix $\Lambda_c$ are the sorted controllability singular values of the unbalanced system such that

$$\Lambda_c = \mathrm{diag}\{\sigma_{c1}, \sigma_{c2}, \ldots, \sigma_{cN}\} \tag{5}$$

with $\sigma_{ci} \geq \sigma_{cj}$ for $i > j$. Large singular values represent linear combinations of states, which are highly controllable, and small singular values represent linear combinations of states, which are slightly controllable. Slightly controllable states may still be important because they can be highly observable.

Similarly, a singular value decomposition is used on $U_c\Lambda_c^{1/2}L_o\Lambda_c^{1/2}U_c^T$ to write

$$U_c\Lambda_c^{\frac{1}{2}}L_o\Lambda_c^{\frac{1}{2}}U_c^T = U_b\Lambda_b^2U_b^T \tag{6}$$

where the elements of the diagonal matrix $\Lambda_b$ are such that

$$\Lambda_b = \mathrm{diag}\{\sigma_{b1}, \sigma_{b2}, \ldots, \sigma_{bN}\} \tag{7}$$

with $\sigma_{bi} \geq \sigma_{bj}$ for $i > j$.

We form a square transformation matrix $T_b$ using

$$T_b = \Lambda_b^{\frac{1}{2}}U_b^TU_c\Lambda_c^{\frac{1}{2}} \tag{8}$$

The inverse can be directly computed using

$$T_b^{-1} = \Lambda_c^{-\frac{1}{2}}U_c^TU_b\Lambda_b^{-\frac{1}{2}} \tag{9}$$

With this transformation we have a balanced system given by $A_b = T_bAT_b^{-1}$, $B_b = T_bB$, and $C_b = CT_b^{-1}$. The controllability and observability gramians become $L_{cb} = T_bL_cT_b^T = \Lambda_b$ and $L_{ob} = T_b^{-T}L_oT_b^{-1} = \Lambda_b$. The elements of the balanced controllability and observability gramian $\Lambda_b$ are the Hankel singular values.

The computation of the inverse of the transformation [Eq. (9)] induces numerical difficulties because it requires inverting 1) the unbalanced controllability singular values $\Lambda_c$ and 2) the Hankel singular values $\Lambda_b$. The numerical conditioning is made poor by the inversion of the small singular values, which we later intend to truncate. A technique, which truncates during the balancing operation, proves to be more numerically robust and allows reduced-order

Received 17 August 2000; revision received 29 May 2001; accepted for publication 30 May 2001. 

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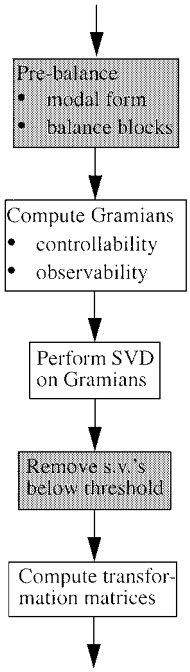


**Fig. 1 Balanced reduction flow diagram. Modifications for increasing the numerical robustness are shaded.**

models of flexible space structures to be computed when standard balanced reduction fails.

## III. Modified Balanced Reduction

To make the balanced reduction more numerically robust, we divide the reduction into two steps: 1) prebalancing and 2) balanced truncation. Figure 1 is a flowchart of the balanced reduction algorithm. Shaded blocks correspond to prebalancing and balanced truncation and are introduced in this Note. The unshaded blocks correspond to the standard balancing algorithm as discussed in Sec. II. Further detail on the algorithm and its application are found in Mallory.[4]

### A. Prebalancing

In the balanced truncation step we will be removing states that we find to be slightly controllable (A dual approach is possible here, where slightly observable states are removed in this step.). The implicit assumption is that those states are not strongly observable. To reduce this possibilty, we apply a prebalancing operation that balances the input/output of each mode.

To apply prebalancing, we first transform the system into a modal form. Complex modes, with denominators $s^2 + 2\zeta\omega s + \omega^2$, are transformed to $2 \times 2$ blocks of the form

$$A_c = \begin{bmatrix} -\zeta\omega & \omega\sqrt{1-\zeta^2} \\ -\omega\sqrt{1-\zeta^2} & -\zeta\omega \end{bmatrix} \tag{10}$$

and real eigenvalues are diagonalized. We then balance the input/output of each complex-conjugate eigenvalue pair and in the case of real eigenvalues each Jordan block, by considering the system part corresponding to those eigenvalues individually. Mode-by-mode prebalancing balances the modal observability and controllability, which reduces the risk of an important mode pairing low controllability with strong observability. The modal approach does carry a risk that linear combinations of modes can be mistreated by the controllability-based balanced truncation step. (To ensure the removed controllable states are not highly observable, a comparison is recommended whereby the model that results from the algorithm as presented can be compared with a model that results from applying the dual algorithm.)

Prebalancing modes that are entirely uncontrollable and/or unobservable are numerically ill-conditioned, but because these modes will later be removed we need not prebalance them.

### B. Balanced Truncation

The balanced truncation removes slightly controllable and slightly observable states as the system is balanced to maintain good numerical conditioning. The method is similar to replacing the inversions required in Eq. (9) with a pseudo-inverse of $T_b$ but allows the designer additional insight and direct manipulation of tolerances for removal of singular values.

We begin by approximating the decomposition of Eq. (4) for the prebalanced system by writing

$$L_c \approx U_c \overline{\Lambda_c} U_c^T \tag{11}$$

where $U_c$ follows from Eq. (4) and $\overline{\Lambda_c}$ is a diagonal matrix of sorted singular values formed by keeping diagonal elements of $\Lambda_c$, which are greater than a specified tolerance and setting other elements to 0

$$\overline{\Lambda_c} = \mathrm{diag}\{\sigma_{c1}, \sigma_{c2}, \ldots, \sigma_{cm}, 0, \ldots, 0\} \tag{12}$$

where $\sigma_{cj} < \mathrm{tol}_c$, $\forall j > m$. This action effectively removes linear combinations of states that are less controllable than the threshold set by $\mathrm{tol}_c$. The prebalancing operation helps to ensure that none of the removed states are highly observable. $\overline{\Lambda_c}^{\dagger}$ is the pseudo-inverse of $\Lambda_c$, formed by

$$\overline{\Lambda_c}^{\dagger} = \mathrm{diag}\{1/\sigma_{c1}, 1/\sigma_{c2}, \ldots, 1/\sigma_{cm}, 0, \ldots, 0\} \tag{13}$$

We perform a similar operation on $\Lambda_b$ by rewriting Eq. (6) as

$$U_c \Lambda_c^{\frac{1}{2}} L_o \Lambda_c^{\frac{1}{2}} U_c^T \approx U_b \overline{\Lambda_b}^2 U_b^T \tag{14}$$

where

$$\overline{\Lambda_b} = \mathrm{diag}\{\sigma_{b1}, \sigma_{b2}, \ldots, \sigma_{bq}, 0 \ldots, 0\} \tag{15}$$

where $\sigma_{bj} < \mathrm{tol}_b$, $\forall j > q$. We form a pseudo-inverse of $\overline{\Lambda_b}$ using

$$\overline{\Lambda_b}^{\dagger} = \mathrm{diag}\{1/\sigma_{b1}, 1/\sigma_{b2}, \ldots, 1/\sigma_{bq}, 0, \ldots, 0\} \tag{16}$$

To reduce the chance of removing a slightly controllable but highly observable state, we choose $\mathrm{tol}_b > \mathrm{tol}_c$ so that $q < m$. In practice, $\mathrm{tol}_b = 100 \times \mathrm{tol}_c$ works well. The choice of these tolerances determines the order of the reduced system. We then form a truncation matrix

$$T_t = [I_{q\times q} 0_{q\times(n-q)}] \tag{17}$$

The balanced truncation transformation is performed with the transformation matrix

$$\overline{T_b} = T_t \overline{\Lambda_b}^{\frac{1}{2}} U_b^T U_c \overline{\Lambda_c}^{\frac{1}{2}} \tag{18}$$

and the (pseudo) inverse transformation

$$\overline{T_b}^{\dagger} = \left(\overline{\Lambda_c}^{\dagger}\right)^{\frac{1}{2}} U_c^T U_b \left(\overline{\Lambda_b}^{\dagger}\right)^{\frac{1}{2}} T_t^T \tag{19}$$

The $n$th order system is reduced to a balanced system of order $q$ by transforming the system as $\overline{A_b} = \overline{T_b} A \overline{T_b}^{\dagger}$, $\overline{B_b} = \overline{T_b} B$, $\overline{C_b} = C\overline{T_b}^{\dagger}$, and $\overline{D_b} = D$. The controllability and observability gramians become $\overline{L_{cb}} = \overline{L_{ob}} = \overline{\Lambda_{br}}$, where

$$\overline{\Lambda_{br}} = \mathrm{diag}\{\sigma_{b1}, \sigma_{b2}, \ldots, \sigma_{bq}\} \tag{20}$$

are the Hankel singular values of the balanced and reduced system. We use the truncation operator $T_t$ to ensure that the $q+1$-th through $n$th states are removed because our singular value truncation in forming $\overline{\Lambda_c}$ and $\overline{\Lambda_b}$ causes those higher states to lose accuracy.

## IV. Example: Space Interferometry Mission FEM

The SIM is an observatory in NASA's Origins program. It is a spaceborne 10-m baseline Michelson optical interferometer. The SIM science goals include furthering astrometry by 1) searching for planetary companions to nearby stars and 2) providing 4 $\mu$arcsec precision absolute star position measures.

Figure 2 is a concept drawing of the SIM Classic spacecraft (The SIM model and ensuing description is for SIM Classic, an early SIM design concept. Current design iterations may result in a spacecraft physically quite different from this. The techniques applied to the SIM classic model are general enough to be applied to future SIM models and other large-scale systems.). The optical side of the spacecraft is formed from a truss structure with seven collector apertures. In the standard observation mode the collectors work in pairs, with two pairs imaging bright guide stars and the third pair imaging the science target. The seventh aperture is redundant. A metrology tower rises from the truss structure. Four beam launchers at the tip of the tower are used to provide four measures of the position of each of the collectors. At the base of the telescope is the spacecraft bus, housing the attitude control system, communication hardware, and electronics.

The FEM model of SIM Classic is utilized in the ensuing design example. The model was created at Jet Propulsion Laboratory, Pasadena, California, using the Integrated Modeling of Optical Systems software toolbox. The model is a stick representation of SIM Classic, with trusses modeled with Bernoulli–Euler beam elements and optics modeled as lumped masses. Modal damping is specified to be 0.1%. The reaction wheel assembly acts on the structure through a modeled six-axis isolator with a 5-Hz corner frequency. Optical ray tracing is used to model the optical sensor and performance output matrices. The raw SIM model has a total of 24 actuators and 39 sensors. A detailed discussion of the model is found in Gutierrez.[5] The raw model is particularly ill-conditioned. Initially the full-order FEM is truncated to a model preserving the first 113 modes with additional dynamics to capture actuator roll off for the large bandwidth actuators (270 states total).

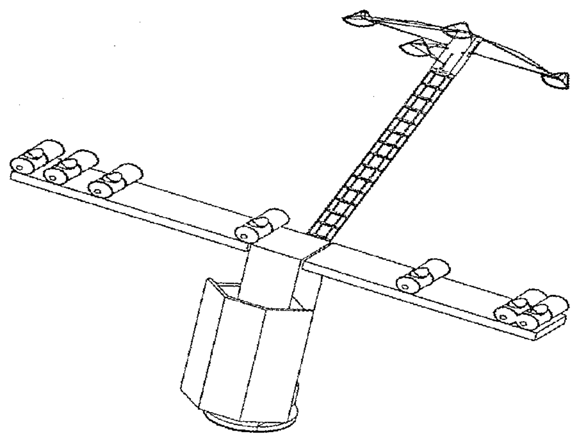

**Fig. 2 SIM Classic: one possible design of the Space Interferometry Mission spacecraft (Graphic courtesy of Jet Propulsion Laboratory, Pasadena, CA).**

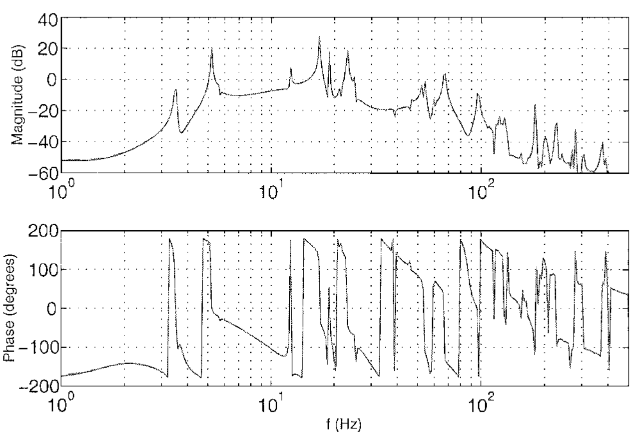


**Fig. 3 Channel transfer function of a balanced model. SIM FEM channel from a reaction wheel disturbance to an internal optical pathlength measure. Phase is wrapped for plotting purposes. The 176-state balanced model (– – –) overlays the 270-state original model (——). Without the numerically robust balancing technique, the SIM model could not be balanced.**

Figure 3 is a plot of the transfer function of a channel from the reaction wheel disturbance to an internal optical pathlength measure for a model of the SIM spacecraft. Conventional balancing routines fail with the SIM model. The numerically robust balancing routine is able to balance the system while removing some uncontrollable/unobservable states. The $\mathrm{tol}_c$ is set to be $10^{-14}$ resulting in $m = 208$; $\mathrm{tol}_b$ is set to be $100 \times \mathrm{tol}_c$ resulting in a $q = 176$ state reduced-order model. In the figure the reduced-order model agrees almost perfectly with the 270-state original model.

Conventional balancing fails for the 270-state full-order SIM model. The condition number for the controllability grammian, and, therefore, $\Lambda_c$ is on the order of $10^{25}$, which exceeds the dynamic range of a 64-bit machine. By applying the numerically improved balancing routine, the condition number of $\overline{\Lambda}_c$ drops to $\sim 10^{10}$ within the machine's precision. For a lightly damped model, like SIM, a modal balancing approach can be effective. For the sample SIM model by reducing the prebalanced model (from Sec. III.A) by extracting $2 \times 2$ blocks with Hankel singular values below a threshold, a 176-state model approaches the model that resulted from the application of the full algorithm. The simplified prebalanced model lost accuracy for frequencies greater than 200 Hz.

## V. Conclusions

The standard balanced reduction methodology has been modified to improve its numerical robustness. Where standard balanced reduction numerically fails, the modifications allow the balanced reduction of prohibitively large FEM models of lightly damped spacecraft. Two modifications to the balanced reduction are introduced: 1) prebalancing the $2 \times 2$ dynamics matrix modal blocks to approximately balance the lightly damped system and 2) simultaneous balancing and truncation to avoid the inaccurate inversion of near-negligible Hankel singular values. The algorithm is demonstrated on a large-order model of the SIM spacecraft. In practice, the numerically robust balancing algorithm has allowed the reduction of spacecraft models for sensitivity analysis and control synthesis and analysis.

## Acknowledgment

The SIM Classic model was provided to the Massachusetts Institute of Technology Space Systems Laboratory by the Jet Propulsion Laboratory for contract SIM 961-123 with technical monitor Sanjay Joshi.

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